Comment on "Extremal-point densities of interface fluctuations in a quenched random medium"

Z. Toroczkai[1] and G. Korniss[2]
[1]*Theoretical Division and Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, New Mexico 87545*
[2]*Department of Physics, Applied Physics, and Astronomy, Rensselaer Polytechnic Institute, Troy, New York 12180*
(Received 5 March 2001; published 21 September 2001)

Lam and Tan [Phys. Rev. E **62**, 6246 (2000)] recently studied the extremal-point densities of interface fluctuations in a quenched random medium. In this Comment we show that their results for systems on a lattice contain algebraic errors leading to invalid conclusions. Further, while most of their calculations for the continuum case are correct, they misinterpret the result to come to an agreement with the (erroneous) lattice calculations. We derive the correct expressions for the lattice, which agree with the correct interpretation of the continuum case.

DOI: 10.1103/PhysRevE.64.048101 PACS number(s): 05.40.−a, 89.20.Ff, 02.70.Rr, 68.35.Ct

The authors of the recent paper, Ref. [1] address a valid question pertaining to the small wavelength physics of a $(1+1)$-dimensional evolving interface in a *quenched random* medium. The authors treat the problem by using a method developed in two earlier papers [2,3], which originally addressed the problem of extremal-point densities for fluctuating interfaces coupled to a time-dependent noise source. The difference between the two problems is that the noise in the quenched random medium has only spatial component, while in the case studied in Refs. [2,3] it also has a temporal dependence. The noise in both cases is assumed to be delta correlated, Gaussian distributed for each component. It is thus natural that the quenched random problem be treated on the same footing as the time-dependent case, by simply dropping the temporal delta function from the covariance of the noise, and adapting our formulas from Refs. [2,3] accordingly [4]. Unfortunately, in this process the authors [1] have made numerous algebraic errors leading to erroneous conclusions. Due to the validity of their original idea, we feel compelled to correct those errors and present the true behavior of their models.

In Sec. II. of [1] the authors attempt to compute the average density of local minima $\langle u\rangle$ of a fluctuating interface in the steady state supported on a one-dimensional lattice. The underlying stochastic process is described by the linear model of molecular beam epitaxy,

$$\partial_t h_i(t)=\nu\nabla^2 h_i(t)-\kappa\nabla^4 h_i(t)+\eta_i\,, \tag{1}$$

where ∇^2 is the discrete Laplacian and η_i is a quenched noise term, delta correlated in space (see [1] for the precise definition of the terms and notations). They derive the structure factor correctly [Eqs. (7) and (14) in [1]], however, they make mistakes in applying the Poisson summation formula to compute the slope-slope correlation function. The steady state structure factor for the *slopes* for this problem is [1]

$$S^{\phi}(k)=\frac{4D}{[1-\cos(k)]\{2\nu+4\kappa[1-\cos(k)]\}^2}. \tag{2}$$

Thus, the slope-slope correlation function will be given by

$$C_L^{\phi}(l)=\frac{1}{L}\sum_{n=1}^{L-1}e^{i(2\pi n/L)l}S^{\phi}\left(\frac{2\pi n}{L}\right)=\frac{4D}{L}\sum_{n=1}^{L-1}\times\frac{e^{i(2\pi n/L)l}}{[1-\cos(2\pi n/L)]\{2\nu+4\kappa[1-\cos(2\pi n/L)]\}^2}. \tag{3}$$

It is rather transparent that their result [Eqs. (20)–(24) in Ref. [1]] for evaluating the above sum cannot be correct: for the general $\nu\neq 0$ case, the $[1-\cos(k)]$ factor in the denominator of the structure factor (2) causes the correlation function $C_L^{\phi}(l)$ to *diverge* as L in the thermodynamic limit. Their expressions [Eqs. (20)–(24)] show that for $L\to\infty$, $C_L^{\phi}(l)$ converges to a *finite* correlation function instead. Consequently, their results for the correlation function ratio [Eq. (25)] and the density of local minima [Eq. (26)] is also erroneous.

While showing that Eqs. (20)–(26) in Ref. [1] cannot be correct is simple, it involves a rather substantial effort to obtain the correct expressions for the above quantities. In order to do so, one can employ the appropriate Poisson summation formula [namely, one for functions with compact support on the real numbers, Eq. (B4) of Refs. [2] or [3]]. Then we obtain

$$C_L^{\phi}(l)=\frac{D}{\nu^2}\left[\frac{L}{6}-A_0(l)-A_1(l)\frac{1}{L}-A_2(l)\frac{Lb^L}{(1-b^L)^2}-A_3(l)\frac{b^L}{(1-b^L)}\right], \tag{4}$$

where

$$A_0(l)=\left(1+\frac{ab^{|l|}}{1+a}\right)|l|+\frac{a(2+a)b^{|l|}}{\sqrt{1-a^2}(1+a)}, \tag{5}$$

$$A_1(l)=\frac{1}{6}-l^2-\frac{a(2+2a+a^2)}{(1+a)(1-a^2)}, \tag{6}$$

$$A_2(l)=\frac{a(2+a)(b^l+b^{-l})}{\sqrt{1-a^2}(1+a)}+\frac{al(b^l-b^{-l})}{1+a}, \tag{7}$$

$$A_3(l)=\frac{a(b^l+b^{-l})}{1+a}, \tag{8}$$

with $a=2\kappa/(\nu+2\kappa)$ and $b=(1-\sqrt{1-a^2})/a$. This result explicitly shows, that the leading term is proportional to L, caused by the divergence for small wave vectors in the structure factor (2). In addition, there is a constant (L independent) term, a uniform power-law correction, and two exponential corrections. Since the density of local minima on a lattice is given by [2,3]

$$\langle u\rangle_L=\frac{1}{2\pi}\arccos\left(\frac{C_L^{\phi}(1)}{C_L^{\phi}(0)}\right), \tag{9}$$

one obtains (after neglecting the exponential corrections for large L)

$$\langle u\rangle_L=\frac{1}{2\pi}\arccos\left(1-\frac{6\sqrt{1-a}}{(1+a)^2}\frac{1}{L}+6\left[1+\frac{6}{(1+a)^3}-\frac{6}{(1+a)}\right]\frac{1}{L^2}+\cdots\right). \tag{10}$$

Thus, for $a<1$ (which is ensured as long as $\nu>0$),

$$\langle u\rangle_L\simeq\frac{\sqrt{3}}{\pi}\frac{(1-a)^{1/4}}{(1+a)^{3/4}}\frac{1}{\sqrt{L}}, \tag{11}$$

or expressing a in terms of ν and κ,

$$\langle u\rangle_L\simeq\frac{\sqrt{3}}{\pi}\frac{(1+2\kappa/\nu)^{1/2}}{(1+4\kappa/\nu)^{3/4}}\frac{1}{\sqrt{L}} \tag{12}$$

in the $L\to\infty$ asymptotic limit. These results show that as long as $\nu>0$, the density of local minima approaches *zero* as $1/\sqrt{L}$, and not a finite nonzero value as claimed in Ref. [1]. In particular, for the pure Edwards-Wilkinson (EW) (or diffusion) dominated regime (described by the $\kappa\to 0$ limit) we obtain

$$\langle u\rangle_L\simeq\frac{\sqrt{3}}{\pi}\frac{1}{\sqrt{L}}\to 0, \tag{13}$$

which contrasts with the erroneous conclusion, $\langle u\rangle_L\to 1/4$, of Ref. [1].

In the Mullins term dominated regime (described by the limit $\nu\to 0$), we have to take into account also the L^{-2} term in Eq. (10) since in this case $a\to 1$. Then we obtain

$$\langle u\rangle_L\simeq\frac{\sqrt{15}}{2\pi}\frac{1}{L}. \tag{14}$$

One can see that for the Mullins regime, the effects of the relaxation are stronger (the density of local minima vanishes faster with system size) than for the diffusion dominated EW regime, as is typical when curvature driven terms are present.

In order to check their results, the authors of Ref. [1] attempt to calculate directly the slope-slope correlation function for the $\kappa=0$ (pure Edwards-Wilkinson) case, Eqs. (27) and (28). Unfortunately, their Eq. (28) does not follow from their Eq. (27). The sum in Eq. (27) is a common expression in statistical field theory on finite lattices. The same summation (with a different prefactor) was calculated and given in Refs. [2] and [3] [see Eq. (31) in these references]. The correct result can also be obtained by setting $\kappa=0$ in Eqs. (4)–(8) above, yielding

$$C_L^{\phi}(l)=\frac{D}{\nu^2}\left[\frac{L}{6}\left(1-\frac{1}{L^2}\right)-|l|\left(1-\frac{|l|}{L}\right)\right]. \tag{15}$$

The correct expression for $\langle u\rangle_L$ is given by Eq. (13), and again it is clear that it approaches zero as $1/\sqrt{L}$ in the asymptotic large-L limit, and not a nonzero constant (1/4), as indicated by Eq. (29) in Ref. [1].

After obtaining all the correct results, a more general consequence for linear growth models with a dynamic exponent z with quenched noise proposed by [1] is clear: the *steady-state* behavior in the presence of a quenched noise and with a dynamic exponent z will be identical to the behavior of the usual time-dependent noise case with a dynamic exponent $2z$. This observation should not come as a surprise: it directly follows form Eq. (7) and (9) of Ref. [1], where the steady-state height structure factor for the quenched medium case is simply proportional to the square of the same quantity for the usual time-dependent noise case. This trivial correspondence will also be explicit in the continuum case as follows below.

In Sec. III of Ref. [1] the authors compute the average density of minima $\langle u\rangle$ for a one-dimensional interface described by the *continuum* Langevin equation

$$\partial_t h(x,t)=-\nu(\nabla^2)^{z/2}h(x,t)+\eta(x), \tag{16}$$

with η being the delta-correlated, time-independent Gaussian noise (see Ref. [1] for the exact definition of the terms and notations). In the *steady-state* regime the *only* difference between their formulas and ours (apart form an overall $1/\nu$ multiplying factor) is that z in our equations has to be replaced by $2z$ to obtain theirs [compare Eq. (47) of Ref. [1] with Eq. (121) of Refs. [2] and [3]]. This trivial change ($z\to 2z$) could lead to the right conclusions immediately for the steady-state behavior. Unfortunately, even in this case one of their major conclusions is incorrect. For $z=2$, below the formula under Eq. (56) in Ref. [1] they argue using $\overline{U}(L,\infty)=[1/\sqrt{2\zeta(2)}](1/\sqrt{La})$ (which is a correct expression) that this is a constant for $La=\text{const}$ (in agreement with their erroneous discrete lattice calculations). This is a misinterpretation of their otherwise algebraically correct result. Note, that in these expressions, L is the physical size of the system, i.e., (the number of lattice sites)$\times a$. The lattice constant a serves as a microscopic cutoff to control the calculations on the continuum. In order to extract *lattice effects* from the continuum approach and to compare it to direct lattice calculations, one has to *fix* the lattice constant a (e.g., $a=1$ for convenience, in which case L becomes the size of the corresponding lattice). Then one immediately sees that the density of local minima goes to zero as $1/\sqrt{L}$ in the large system-size limit in full agreement with the correct lattice results, Eq. (13) above. Their agreement between their discrete and the continuum calculations is a result of using an incorrect interpretation of the latter to match the erroneous algebra of the former.

In Sec. III B (''Scaling regime'') of Ref. [1] the authors calculate the temporal behavior of the local minimum density and the related partition functions, based on the formalism developed in Refs. [2,3]. While most of the expressions obtained by the authors in Ref. [1] are correct in principle, there are numerous typographical errors in this section. Due to the scope of this paper we limit ourselves to list only a few of those, which we believe are serious enough to hinder clarity and understanding for the general reader.

The correct argument of the gamma function in Eq. (61) of Ref. [1] should read $(m-2z+1)/z$, as shown correctly in Eq. (62). These expressions are valid for *both* $z<(m+1)/2$ and $z>(m+1)/2$ regimes, not only $z<(m+1)/2$, as the authors mention. Under Eq. (64) of Ref. [1] the authors mention that C is ''a'' constant. We note that $C=\mathcal{C}-\ln 2$ where $\mathcal{C}$ is *the* Euler constant. In Ref. [1] Eq. (67) is missing a $[1-E_m]$ factor and there should be a minus sign in front of the exponent of 2; in Eq. (68) the factor $(1-E_4)$ should be in the numerator and there should be a factor 4 (as opposed to 2) in front of $\Gamma(3/2)$ in the denominator; Eq. (69) should not have a factor 2 in the numerator.

Some of the conclusions they draw from Eqs. (68–70) for the $z>5/2$ also contain errors. Correctly they should read: $U_q(L,t)\sim t^{-[2-q(2z-5)]/(2z)}$, $\overline{U}(L,t)\sim t^{-1/z}$ (the latter they obtained correctly), and $K(L,t)\sim t^{(2z-5)/(2z)}$. The statement in the second sentence after Eq. (70) ''... with $z=4$, the density of local minima decreases in time as $t^{-1/2}$...'' is incorrect, since $\overline{U}(L,t)\sim t^{-1/z}=t^{-1/4}$.

For the case $z=5/2$ we would like to point out that the otherwise correct results, Eqs. (71)–(74) of Ref. [1], contain no logarithmic ''corrections'' as referred to by the authors after Eq. (74). Logarithmic scaling is the *leading* behavior itself and not a correction: $U_q(L,t)\sim t^{-2/5}(\ln t)^{(q+1)/2}$, $\overline{U}(L,t)\sim t^{-2/5}\sqrt{\ln t}$, and $K(L,t)\sim\sqrt{\ln t}$.

For the regime $3/2<z<5/2$ and also the rest of the cases the authors do not attempt to show the explicit scaling behavior. The temporal scaling behavior for the three quantities they compute are: $U_q(L,t)\sim t^{-(2z-3)/(2z)}$, and thus $\overline{U}(L,t)\sim t^{-(2z-3)/(2z)}$, and $K(L,t)$ becomes a constant in leading order (for fixed a) with a next-term temporal correction of $t^{-(5-2z)/z}$. In particular for the pure Edwards-Wilkinson case ($z=2$) the density of local minima vanishes as $t^{-1/4}$. For this $z=2$ case they again attempt to extract lattice effects from the continuum approach to compare them to the direct lattice calculations. In order to do so, they use the $\xi\to\infty$, $a\to 0$ ($\xi a=$const) limit. This limit has nothing to do with extracting lattice results (for which one simply has to keep a fixed). The only reason why their ill-defined limit is in agreement with their lattice result, because their lattice calculation was erroneous to begin with.

For the $z=3/2$ case the temporal scaling behavior (leading order) is as follows: $U_q(L,t)\sim(\ln t)^{-1/2}$, and $\overline{U}(L,t)\sim(\ln t)^{-1/2}$.

In the regime $1<z<3/2$ the quantities are led by time-independent terms (constants) followed by time-dependent corrections: for both $U_q(L,t)$ and $\overline{U}(L,t)$ the *correction* terms have a temporal dependence of $t^{-(3-2z)/(2z)}$.

In summary, the authors in Ref. [1] address a valid question, using the formalism and language (at points taken literally) developed in Refs. [2,3]. However, their lattice calculations are not sound algebraically and the results are incorrect. Then after carrying out a continuum-based approach and using an incorrect limit to interpret their otherwise correct formulas, they ''manage'' to find agreement between the two approaches.

Z.T. was supported by the Department of Energy, under Contract No. W-7405-ENG-36.

[1] P.-M. Lam and S. Tan, Phys. Rev. E **62**, 6246 (2000).
[2] Z. Toroczkai, G. Korniss, S. Das Sarma, and R.K.P. Zia, Phys. Rev. E **62**, 276 (2000).
[3] Z. Toroczkai, G. Korniss, S. Das Sarma, and R.K.P. Zia, http://xxx.lanl.gov/abs/cond-mat/0002143.
[4] For the continuum case Rice [5] obtained analytic expressions for the extremal-point densities (in the context of noise effects on telephone transmission) without using the path integral and ''partition function'' formalism.
[5] S.O. Rice, Bell Syst. Tech. J. **23**, 282 (1944).